Regular and Context-Free Pattern Languages Over Small Alphabets

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DLT 2012

Pattern languages 1/2

Σ: Finite alphabet of terminal symbols (e. g. $\Sigma = \{a, b, c, d\}$).

$X = \{x_1, x_2, x_3, \ldots\}$: Infinite alphabet of variables.

A word $\alpha \in (\Sigma \cup X)^+$ is called a *pattern*.

Basic Definitions

Morphism: Mapping $h : \Gamma_1^* \to \Gamma_2^*$ with $h(x \cdot y) = h(x) \cdot h(y)$;
h is *nonerasing* iff, for every $a \in \Gamma_1$, $h(a) \neq \varepsilon$.

Substitution: Morphism $h : (\Sigma \cup X)^* \to \Sigma^*$ with $h(a) = a$ for all $a \in \Sigma$.

$L_{\mathrm{E},\Sigma}(\alpha) := \{h(\alpha) \mid h \text{ is a substitution}\}$.

$L_{\mathrm{NE},\Sigma}(\alpha) := \{h(\alpha) \mid h \text{ is nonerasing substitution}\}$.

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$L_{\mathrm{NE},\{\mathtt{a},\mathtt{b},\mathtt{c}\}}(\alpha) = \{w \mid w = u \, \mathtt{aa} \, v \, u \, v \, \mathtt{cb} \, u, \; u, v \in \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}^+\}.$

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$\mathtt{ccbaaccbcbccb} \notin L_{\mathrm{NE},\{\mathtt{a},\mathtt{b},\mathtt{c}\}}(\alpha)$

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$\mathtt{ccbaaccbcbccb} \notin L_{\mathrm{NE},\{\mathtt{a},\mathtt{b},\mathtt{c}\}}(\alpha)$
$\mathtt{ccbaaccbcbccb} \in L_{\mathrm{E},\{\mathtt{a},\mathtt{b},\mathtt{c}\}}(\alpha)$

Notation

- var(α): Set of variables occurring in α.
  E. g. $\mathrm{var}(x_1\, ab\, x_2\, ba\, x_1\, x_2\, c\, x_3) = \{x_1, x_2, x_3\}$.
- $|\alpha|_{x_i}$: Number of occurrences of variable x_i in α.
  E. g. $|x_1\, ab\, x_2\, ba\, x_1\, x_2\, c\, x_3|_{x_2} = 2$.

- REG := regular languages,
- CF := context-free languages,
- CS := context-sensitive languages,
- $Z \in \{E, NE\}$, $Z\text{-PAT}_\Sigma$ = Z-Pattern Languages (w. r. t. Σ).

Pattern Languages and the Chomsky Hierachy

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  - $Z\text{-PAT}_\Sigma \,\#\, \text{REG}$, $Z\text{-PAT}_\Sigma \,\#\, \text{CF}$,
  - $Z\text{-PAT}_\Sigma \cap \text{REG} \neq \emptyset$, $Z\text{-PAT}_\Sigma \cap \text{CF} \neq \emptyset$.

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  - $Z\text{-PAT}_\Sigma \cap \text{REG} \neq \emptyset$, $Z\text{-PAT}_\Sigma \cap \text{CF} \neq \emptyset$.
- The sets $Z\text{-PAT}_\Sigma \cap \text{REG}$ and $Z\text{-PAT}_\Sigma \cap \text{CF}$ are surprisingly difficult to characterise.

Notation

$$L_{\mathsf{E},\Sigma}(\alpha) \in \mathsf{REG} \Leftrightarrow \alpha \in \mathsf{REG}\ (\mathsf{E},\ \Sigma),$$

$$L_{\mathsf{Z},\Sigma}(\alpha) \in \mathsf{REG},\ \text{for every}\ \mathsf{Z} \in \{\mathsf{E}, \mathsf{NE}\} \Leftrightarrow \alpha \in \mathsf{REG}\ (\mathsf{Z},\ \Sigma).$$

Examples

$x_1 \, \mathtt{a} \, x_2 \, \mathtt{b} \, x_3 \, \mathtt{c} \, x_4 \in \mathrm{REG}\,(\mathrm{NE}, \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}),$

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- $x_1\, x_2\, x_3\, \mathtt{a}\, x_2\, x_4\, x_4\, x_5\, \mathtt{a}\, x_6\, x_5\, x_7 \in \text{REG}\,(\text{E}, \{\mathtt{a}, \mathtt{b}\})$,
- $x_1\, x_2\, x_3\, \mathtt{a}\, x_2\, x_4\, x_4\, x_5\, \mathtt{b}\, x_6\, x_5\, x_7 \in \text{REG}\,(\text{E}, \{\mathtt{a}, \mathtt{b}, \mathtt{c}\})$,

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- $\mathtt{a} \, x_1 \, \mathtt{a} \, x_2 \, \mathtt{a} \, x_1 \, \mathtt{a} \, x_3 \in \mathrm{CF} \setminus \mathrm{REG}, (\mathrm{Z}, \{\mathtt{a}, \mathtt{b}\})$,
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- $\mathtt{a}\,x_1\,\mathtt{a}\,x_2\,\mathtt{a}\,x_1\,\mathtt{a}\,x_3 \in \text{CF} \setminus \text{REG}, (\text{Z}, \{\mathtt{a}, \mathtt{b}\})$,
- $x_1\,\mathtt{a}\,x_2\,\mathtt{b}\,x_3\,\mathtt{a}\,x_2\,\mathtt{b}\,x_4 \in \text{CF} \setminus \text{REG}\,(\text{Z}, \{\mathtt{a}, \mathtt{b}, \mathtt{c}\})$.

$x_1\mathtt{a}x_2{x_3}^2\mathtt{b}x_4x_3x_5x_6{x_7}^2x_8x_9x_5x_3\mathtt{a}x_4x_5x_4x_9x_{10}\mathtt{b}x_{11}\mathtt{a}x_{12}\mathtt{b}x_{13}\mathtt{a}x_{14}x_{15}\mathtt{b}{x_{15}}^2{x_{16}}^2\mathtt{b}x_{17}$
$\in \text{REG}\,(\text{E}, \{\mathtt{a}, \mathtt{b}\})$.

Regular and Block-Regular Patterns 1/2

- *Regular patterns*
  - A pattern is a *regular* pattern iff every variable has only one occurrence,
  - e. g., $x_1 \, \mathtt{a} \, x_2 \, \mathtt{b} \, x_3 \, \mathtt{c} \, x_4$.
  - $Z \in \{E, NE\}$, $Z\text{-PAT}_{\Sigma,\text{reg}}$ = Z-Pattern Languages (w. r. t. Σ) defined by regular patterns.

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- *Block-regular patterns*
  - A pattern is a *block-regular* pattern iff every *variable block* contains a variable with only one occurrence in the whole pattern,
  - e. g., $x_1 x_2 \, \mathtt{a} \, x_1 x_3 x_4 x_1 x_4 \, \mathtt{b} \, x_5 x_6 x_4 \, \mathtt{c} \, x_7$,
  - $\mathrm{Z} \in \{\mathrm{E}, \mathrm{NE}\}$, $\text{Z-PAT}_{\Sigma,\text{b-reg}}$ = Z-Pattern Languages (w. r. t. Σ) defined by block-regular patterns.

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- If $|\Sigma| = 1$ and $\text{Z} \in \{\text{E}, \text{NE}\}$, then $\text{Z-PAT}_\Sigma = (\text{Z-PAT}_\Sigma \cap \text{REG})$.

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- Jain et al. show that a *terminal-free* pattern α describes a regular (context-free) E-pattern language iff at least one variable in α has only one occurrence.
- Hard cases:
  - NE case in general,
  - E case for alphabets of size 2 and 3.

Repeated Variables

Theorem (Jain et al., 2010)

Let $|\Sigma| \geq 2$ *and let* $\alpha \in X^+$. *If, for every* $x \in \text{var}(\alpha)$, $|\alpha|_x \geq 2$, *then* $\alpha \notin \text{CF}\,(\text{E}, \Sigma)$ *and* $\alpha \notin \text{REG}\,(\text{NE}, \Sigma)$.

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Theorem

Let $|\Sigma| \geq 2$ *and let* $\alpha \in (\Sigma \cup X)^+$. *If, for every* $x \in \text{var}(\alpha)$, $|\alpha|_x \geq 2$, *then* $\alpha \notin \text{REG}$ *(Z, Σ).*

Theorem

Let $|\Sigma| \geq 3$ *and let* $\alpha \in (\Sigma \cup X)^+$. *If, for every* $x \in \text{var}(\alpha)$, $|\alpha|_x \geq 2$, *then* $\alpha \notin \text{CF}$ *(Z, Σ).*

Spread Variable Blocks

Theorem

Let Σ *be terminal alphabets with* $|\Sigma| \leq 3$ *and let* $\mathsf{Z} \in \{\mathsf{E}, \mathsf{NE}\}$*. Let* α *be a pattern with*

$$\alpha = \beta \cdot d \cdot \gamma \cdot d' \cdot \delta, \text{ where } d, d' \in \Sigma, \gamma \in X^+.$$

If $\mathrm{var}(\gamma) \subseteq \mathrm{var}(\beta) \cup \mathrm{var}(\delta)$*, then* $\alpha \notin \mathrm{REG}$ *(*Z*,* Σ'*).*

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We now focus on

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- patterns with variables that are not repeated.

Example

E case:

$$|\Sigma| = 2: \quad x_1 \; \mathtt{a} \; x_2 \; x_2 \; \mathtt{a} \; x_3 \quad \in \text{REG},$$

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Example

E case:

$$|\Sigma| = 2: \quad x_1 \; \mathtt{a} \; x_2 \; x_2 \; \mathtt{a} \; x_3 \quad \in \text{REG}, \; x_1 \; \mathtt{a(bb)^*a} \; x_3$$
$$|\Sigma| \geq 3: \quad x_1 \; \mathtt{a} \; x_2 \; x_2 \; \mathtt{a} \; x_3 \quad \notin \text{REG},$$
$$|\Sigma| = 2: \quad x_1 \; \mathtt{a} \; x_2 \; x_2 \; \mathtt{b} \; x_3 \quad \in \text{REG}, \; x_1 \; \mathtt{ab} \; x_3$$
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Example

E, NE case , $|\Sigma| = 2$: $x_1 \, \mathtt{a} \, x_2 \, \mathtt{a} \, x_2 \, \mathtt{a} \, x_3 \notin \mathrm{REG}$,

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$$\begin{aligned}
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\text{E, NE case}, |\Sigma| \geq 3 &: \quad x_1\ \mathtt{a}\ x_2\ \mathtt{b}\ x_2\ \mathtt{a}\ x_3 \quad \notin \text{REG}.
\end{aligned}$$

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$$\text{E, NE case}, |\Sigma| = 2: \quad x_1 \, \mathtt{a} \, x_2 \, \mathtt{a} \, x_2 \, \mathtt{a} \, x_3 \quad \notin \text{REG},$$

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$$\text{NE case}, |\Sigma| = 2: \quad x_1 \, \mathtt{a} \, x_2 \, \mathtt{b} \, x_2 \, \mathtt{a} \, x_3 \quad \notin \text{REG},$$

$$\text{E, NE case}, |\Sigma| \geq 3: \quad x_1 \, \mathtt{a} \, x_2 \, \mathtt{b} \, x_2 \, \mathtt{a} \, x_3 \quad \notin \text{REG}.$$

Example

$$\mathrm{E}, \mathrm{NE}\ \ \text{case}\ , |\Sigma| = 2 : \quad x_1\ \mathtt{a}\ x_2\ x_2\ \mathtt{a}\ x_3\ x_3\ \mathtt{a}\ x_4 \quad \notin \mathrm{REG},$$

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$$\text{E}, \text{NE case}, |\Sigma| = 2: \quad x_1 \; \mathtt{a} \; x_2 \; x_2 \; \mathtt{a} \; x_3 \; x_3 \; \mathtt{a} \; x_4 \quad \notin \text{REG},$$

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$$\text{NE case}, |\Sigma| = 2: \quad x_1\ \mathtt{a}\ x_2\ x_2\ \mathtt{b}\ x_3\ x_3\ \mathtt{a}\ x_4 \quad \notin \text{REG},$$
$$\text{E case}, |\Sigma| \geq 3: \quad x_1\ \mathtt{a}\ x_2\ x_2\ \mathtt{b}\ x_3\ x_3\ \mathtt{a}\ x_4 \quad \notin \text{REG},$$
$$\text{E case}, |\Sigma| = 2: \quad x_1\ \mathtt{a}\ x_2\ x_2\ \mathtt{a}\ x_3\ x_3\ \mathtt{b}\ x_4 \quad \in \text{REG},$$

Example

$$\text{E, NE case}, |\Sigma| = 2: \quad x_1\ \mathtt{a}\ x_2\ x_2\ \mathtt{a}\ x_3\ x_3\ \mathtt{a}\ x_4 \quad \notin \text{REG},$$
$$\text{E case}, |\Sigma| = 2: \quad x_1\ \mathtt{a}\ x_2\ x_2\ \mathtt{b}\ x_3\ x_3\ \mathtt{a}\ x_4 \quad \in \text{REG},$$

$$\text{NE case}, |\Sigma| = 2: \quad x_1\ \mathtt{a}\ x_2\ x_2\ \mathtt{b}\ x_3\ x_3\ \mathtt{a}\ x_4 \quad \notin \text{REG},$$
$$\text{E case}, |\Sigma| \geq 3: \quad x_1\ \mathtt{a}\ x_2\ x_2\ \mathtt{b}\ x_3\ x_3\ \mathtt{a}\ x_4 \quad \notin \text{REG},$$
$$\text{E case}, |\Sigma| = 2: \quad x_1\ \mathtt{a}\ x_2\ x_2\ \mathtt{a}\ x_3\ x_3\ \mathtt{b}\ x_4 \quad \in \text{REG},$$

$$\text{NE case}, |\Sigma| = 2: \quad x_1\ \mathtt{a}\ x_2\ x_2\ \mathtt{a}\ x_3\ x_3\ \mathtt{b}\ x_4 \quad \notin \text{REG},$$

Example

$$\text{E, NE case}, |\Sigma| = 2: \quad x_1\ \mathtt{a}\ x_2\ x_2\ \mathtt{a}\ x_3\ x_3\ \mathtt{a}\ x_4 \quad \notin \text{REG},$$
$$\text{E case}, |\Sigma| = 2: \quad x_1\ \mathtt{a}\ x_2\ x_2\ \mathtt{b}\ x_3\ x_3\ \mathtt{a}\ x_4 \quad \in \text{REG},$$

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$$\text{E case}, |\Sigma| \geq 3: \quad x_1\ \mathtt{a}\ x_2\ x_2\ \mathtt{b}\ x_3\ x_3\ \mathtt{a}\ x_4 \quad \notin \text{REG},$$
$$\text{E case}, |\Sigma| = 2: \quad x_1\ \mathtt{a}\ x_2\ x_2\ \mathtt{a}\ x_3\ x_3\ \mathtt{b}\ x_4 \quad \in \text{REG},$$

$$\text{NE case}, |\Sigma| = 2: \quad x_1\ \mathtt{a}\ x_2\ x_2\ \mathtt{a}\ x_3\ x_3\ \mathtt{b}\ x_4 \quad \notin \text{REG},$$
$$\text{E case}, |\Sigma| \geq 3: \quad x_1\ \mathtt{a}\ x_2\ x_2\ \mathtt{a}\ x_3\ x_3\ \mathtt{b}\ x_4 \quad \notin \text{REG},$$

Example

$$\text{E, NE case}, |\Sigma| = 2: \quad x_1 \, \mathtt{a} \, x_2 \, x_2 \, \mathtt{a} \, x_3 \, x_3 \, \mathtt{a} \, x_4 \quad \notin \text{REG},$$

$$\text{E case}, |\Sigma| = 2: \quad x_1 \, \mathtt{a} \, x_2 \, x_2 \, \mathtt{b} \, x_3 \, x_3 \, \mathtt{a} \, x_4 \quad \in \text{REG},$$

$$\text{NE case}, |\Sigma| = 2: \quad x_1 \, \mathtt{a} \, x_2 \, x_2 \, \mathtt{b} \, x_3 \, x_3 \, \mathtt{a} \, x_4 \quad \notin \text{REG},$$

$$\text{E case}, |\Sigma| \geq 3: \quad x_1 \, \mathtt{a} \, x_2 \, x_2 \, \mathtt{b} \, x_3 \, x_3 \, \mathtt{a} \, x_4 \quad \notin \text{REG},$$

$$\text{E case}, |\Sigma| = 2: \quad x_1 \, \mathtt{a} \, x_2 \, x_2 \, \mathtt{a} \, x_3 \, x_3 \, \mathtt{b} \, x_4 \quad \in \text{REG},$$

$$\text{NE case}, |\Sigma| = 2: \quad x_1 \, \mathtt{a} \, x_2 \, x_2 \, \mathtt{a} \, x_3 \, x_3 \, \mathtt{b} \, x_4 \quad \notin \text{REG},$$

$$\text{E case}, |\Sigma| \geq 3: \quad x_1 \, \mathtt{a} \, x_2 \, x_2 \, \mathtt{a} \, x_3 \, x_3 \, \mathtt{b} \, x_4 \quad \notin \text{REG},$$

$$L_{\text{E},\{\mathtt{a},\mathtt{b}\}}(x_1 d_1 x_2 x_2 d_2 x_3 x_3 d_3 x_4) \notin \text{REG iff } d_1 = d_2 = d_3.$$

Example

$$\begin{array}{rl}
\text{E, NE case}, |\Sigma| = 2: & x_1\ \mathtt{a}\ x_2\ x_2\ \mathtt{a}\ x_3\ x_3\ \mathtt{a}\ x_4 \quad \notin \text{REG}, \\
\text{E case}, |\Sigma| = 2: & x_1\ \mathtt{a}\ x_2\ x_2\ \mathtt{b}\ x_3\ x_3\ \mathtt{a}\ x_4 \quad \in \text{REG}, \\
 & x_1 \cdot \mathtt{a} \cdot (\mathtt{bb})^* \cdot \mathtt{b} \cdot \mathtt{a} \cdot x_4 \\
\text{NE case}, |\Sigma| = 2: & x_1\ \mathtt{a}\ x_2\ x_2\ \mathtt{b}\ x_3\ x_3\ \mathtt{a}\ x_4 \quad \notin \text{REG}, \\
\text{E case}, |\Sigma| \geq 3: & x_1\ \mathtt{a}\ x_2\ x_2\ \mathtt{b}\ x_3\ x_3\ \mathtt{a}\ x_4 \quad \notin \text{REG}, \\
\text{E case}, |\Sigma| = 2: & x_1\ \mathtt{a}\ x_2\ x_2\ \mathtt{a}\ x_3\ x_3\ \mathtt{b}\ x_4 \quad \in \text{REG}, \\
 & x_1 \cdot \mathtt{a} \cdot (\mathtt{bb})^* \cdot \mathtt{a} \cdot \mathtt{b} \cdot x_4 \\
\text{NE case}, |\Sigma| = 2: & x_1\ \mathtt{a}\ x_2\ x_2\ \mathtt{a}\ x_3\ x_3\ \mathtt{b}\ x_4 \quad \notin \text{REG}, \\
\text{E case}, |\Sigma| \geq 3: & x_1\ \mathtt{a}\ x_2\ x_2\ \mathtt{a}\ x_3\ x_3\ \mathtt{b}\ x_4 \quad \notin \text{REG},
\end{array}$$

$L_{\text{E},\{\mathtt{a},\mathtt{b}\}}(x_1 d_1 x_2 x_2 d_2 x_3 x_3 d_3 x_4) \notin \text{REG}$ iff $d_1 = d_2 = d_3$.

Two Basic Lemmas

Lemma 1: $L_{\mathrm{E},\{\mathtt{a},\mathtt{b}\}}(\beta \cdot y \cdot \beta' \cdot \mathtt{a} \cdot \gamma \cdot \mathtt{b} \cdot \delta' \cdot z \cdot \delta) = L_{\mathrm{E},\{\mathtt{a},\mathtt{b}\}}(\beta \cdot y \cdot \mathtt{ab} \cdot z \cdot \delta)$,

- $\beta, \delta \in (\{\mathtt{a}, \mathtt{b}\} \cup X)^*$,
- $\beta', \gamma, \delta' \in X^*$,
- $y, z \in X$, $|\alpha|_y = |\alpha|_z = 1$,
- $\mathrm{var}(\beta' \cdot \gamma \cdot \delta') \cap \mathrm{var}(\beta \cdot \delta) = \emptyset$,

Two Basic Lemmas

Lemma 1: $L_{E,\{a,b\}}(\beta \cdot y \cdot \beta' \cdot a \cdot \gamma \cdot b \cdot \delta' \cdot z \cdot \delta) = L_{E,\{a,b\}}(\beta \cdot y \cdot ab \cdot z \cdot \delta)$,

Lemma 2: $L_{E,\{a,b\}}(\beta \cdot y \cdot \beta' \cdot a \cdot \gamma \cdot a \cdot \delta' \cdot z \cdot \delta) = L_{E,\{a,b\}}(\beta \cdot y \cdot a(bb)^* a \cdot z \cdot \delta)$,

- $\beta, \delta \in (\{a, b\} \cup X)^*$,
- $\beta', \gamma, \delta' \in X^*$,
- $y, z \in X$, $|\alpha|_y = |\alpha|_z = 1$,
- $\text{var}(\beta' \cdot \gamma \cdot \delta') \cap \text{var}(\beta \cdot \delta) = \emptyset$,
- there exists a $z' \in \text{var}(\gamma)$ with $|\alpha|_{z'} = 2$,
- $|\gamma|_x$ is even, $x \in \text{var}(\gamma)$.

Example

$$x_1\,\mathtt{a}\,x_2\,x_3^2\,\mathtt{b}\,x_4\,x_3\,x_5\,x_6\,x_7^2\,x_8\,x_9\,x_5\,x_3\,\mathtt{a}\,x_4\,x_5\,x_4\,x_9\,x_{10}\,\mathtt{b}\,x_{11}\,\mathtt{a}\,x_{12}\,\mathtt{b}\,x_{13}\,\mathtt{a}\,x_{14}\,x_{15}\,\mathtt{b}\,x_{15}^2\,x_{16}^2\,\mathtt{b}\,x_{17}$$

Example

$$x_1 \, a \, x_2 \, x_3^2 \, b \, x_4 \, x_3 \, x_5 \, x_6 \, x_7^2 \, x_8 \, x_9 \, x_5 \, x_3 \, a \, x_4 \, x_5 \, x_4 \, x_9 \, x_{10} \, b \, x_{11} \, a \, x_{12} \, b \, x_{13} \, a \, x_{14} \, x_{15} \, b \, x_{15}^2 \, x_{16}^2 \, b \, x_{17}$$

Example

$$x_1 \mathtt{a} x_2 x_3^2 \mathtt{b} x_4 x_3 x_5 x_6 x_7^2 x_8 x_9 x_5 x_3 \mathtt{a} x_4 x_5 x_4 x_9 x_{10} \mathtt{b} x_{11} \mathtt{a} x_{12} \mathtt{b} x_{13} \mathtt{a} x_{14} x_{15} \mathtt{b} x_{15}^2 x_{16}^2 \mathtt{b} x_{17}$$

$$x_1 \mathtt{a}\mathtt{b} x_6 x_7^2 x_8 x_9 x_5 x_3 \mathtt{a} x_4 x_5 x_4 x_9 x_{10} \mathtt{b} x_{11} \mathtt{a} x_{12} \mathtt{b} x_{13} \mathtt{a} x_{14} x_{15} \mathtt{b} x_{15}^2 x_{16}^2 \mathtt{b} x_{17}$$

Example

$$x_1\,\mathtt{a}\,x_2\,x_3^2\,\mathtt{b}\,x_4\,x_3\,x_5\,x_6\,x_7^2\,x_8\,x_9\,x_5\,x_3\,\mathtt{a}\,x_4\,x_5\,x_4\,x_9\,x_{10}\,\mathtt{b}\,x_{11}\,\mathtt{a}\,x_{12}\,\mathtt{b}\,x_{13}\,\mathtt{a}\,x_{14}\,x_{15}\,\mathtt{b}\,x_{15}^2\,x_{16}^2\,\mathtt{b}\,x_{17}$$

$$x_1\,\mathtt{a}\mathtt{b}\,x_6\,x_7^2\,x_8\,x_9\,x_5\,x_3\,\mathtt{a}\,x_4\,x_5\,x_4\,x_9\,x_{10}\,\mathtt{b}\,x_{11}\,\mathtt{a}\,x_{12}\,\mathtt{b}\,x_{13}\,\mathtt{a}\,x_{14}\,x_{15}\,\mathtt{b}\,x_{15}^2\,x_{16}^2\,\mathtt{b}\,x_{17}$$

Example

$$x_1\,\mathtt{a}\,x_2\,x_3^2\,\mathtt{b}\,x_4\,x_3\,x_5\,x_6\,x_7^2\,x_8\,x_9\,x_5\,x_3\,\mathtt{a}\,x_4\,x_5\,x_4\,x_9\,x_{10}\,\mathtt{b}\,x_{11}\,\mathtt{a}\,x_{12}\,\mathtt{b}\,x_{13}\,\mathtt{a}\,x_{14}\,x_{15}\,\mathtt{b}\,x_{15}^2\,x_{16}^2\,\mathtt{b}\,x_{17}$$

$$x_1\,\mathtt{a}\mathtt{b}\,x_6\,x_7^2\,x_8\,x_9\,x_5\,x_3\,\mathtt{a}\,x_4\,x_5\,x_4\,x_9\,x_{10}\,\mathtt{b}\,x_{11}\,\mathtt{a}\,x_{12}\,\mathtt{b}\,x_{13}\,\mathtt{a}\,x_{14}\,x_{15}\,\mathtt{b}\,x_{15}^2\,x_{16}^2\,\mathtt{b}\,x_{17}$$

Example

$x_1 \mathtt{a} x_2 x_3^2 \mathtt{b} x_4 x_3 x_5 x_6 x_7^2 x_8 x_9 x_5 x_3 \mathtt{a} x_4 x_5 x_4 x_9 x_{10} \mathtt{b} x_{11} \mathtt{a} x_{12} \mathtt{b} x_{13} \mathtt{a} x_{14} x_{15} \mathtt{b} x_{15}^2 x_{16}^2 \mathtt{b} x_{17}$

$x_1 \mathtt{ab} x_6 x_7^2 x_8 x_9 x_5 x_3 \mathtt{a} x_4 x_5 x_4 x_9 x_{10} \mathtt{b} x_{11} \mathtt{a} x_{12} \mathtt{b} x_{13} \mathtt{a} x_{14} x_{15} \mathtt{b} x_{15}^2 x_{16}^2 \mathtt{b} x_{17}$

$x_1 \mathtt{ab} x_6 x_7^2 x_8 \mathtt{ab} x_{11} \mathtt{a} x_{12} \mathtt{b} x_{13} \mathtt{a} x_{14} x_{15} \mathtt{b} x_{15}^2 x_{16}^2 \mathtt{b} x_{17}$

Example

$$x_1 \mathtt{a} x_2 x_3^2 \mathtt{b} x_4 x_3 x_5 x_6 x_7^2 x_8 x_9 x_5 x_3 \mathtt{a} x_4 x_5 x_4 x_9 x_{10} \mathtt{b} x_{11} \mathtt{a} x_{12} \mathtt{b} x_{13} \mathtt{a} x_{14} x_{15} \mathtt{b} x_{15}^2 x_{16}^2 \mathtt{b} x_{17}$$

$$x_1 \mathtt{a}\mathtt{b} x_6 x_7^2 x_8 x_9 x_5 x_3 \mathtt{a} x_4 x_5 x_4 x_9 x_{10} \mathtt{b} x_{11} \mathtt{a} x_{12} \mathtt{b} x_{13} \mathtt{a} x_{14} x_{15} \mathtt{b} x_{15}^2 x_{16}^2 \mathtt{b} x_{17}$$

$$x_1 \mathtt{a}\mathtt{b} x_6 x_7^2 x_8 \mathtt{a}\mathtt{b} x_{11} \mathtt{a} x_{12} \mathtt{b} x_{13} \mathtt{a} x_{14} x_{15} \mathtt{b} x_{15}^2 x_{16}^2 \mathtt{b} x_{17}$$

Example

$$x_1 \mathtt{a} x_2 x_3^2 \mathtt{b} x_4 x_3 x_5 x_6 x_7^2 x_8 x_9 x_5 x_3 \mathtt{a} x_4 x_5 x_4 x_9 x_{10} \mathtt{b} x_{11} \mathtt{a} x_{12} \mathtt{b} x_{13} \mathtt{a} x_{14} x_{15} \mathtt{b} x_{15}^2 x_{16}^2 \mathtt{b} x_{17}$$

$$x_1 \mathtt{a}\mathtt{b} x_6 x_7^2 x_8 x_9 x_5 x_3 \mathtt{a} x_4 x_5 x_4 x_9 x_{10} \mathtt{b} x_{11} \mathtt{a} x_{12} \mathtt{b} x_{13} \mathtt{a} x_{14} x_{15} \mathtt{b} x_{15}^2 x_{16}^2 \mathtt{b} x_{17}$$

$$x_1 \mathtt{a}\mathtt{b} x_6 x_7^2 x_8 \mathtt{a}\mathtt{b} x_{11} \mathtt{a} x_{12} \mathtt{b} x_{13} \mathtt{a} x_{14} x_{15} \mathtt{b} x_{15}^2 x_{16}^2 \mathtt{b} x_{17}$$

Example

$$x_1 \mathtt{a} x_2 x_3^2 \mathtt{b} x_4 x_3 x_5 x_6 x_7^2 x_8 x_9 x_5 x_3 \mathtt{a} x_4 x_5 x_4 x_9 x_{10} \mathtt{b} x_{11} \mathtt{a} x_{12} \mathtt{b} x_{13} \mathtt{a} x_{14} x_{15} \mathtt{b} x_{15}^2 x_{16}^2 \mathtt{b} x_{17}$$

$$x_1 \mathtt{ab} x_6 x_7^2 x_8 x_9 x_5 x_3 \mathtt{a} x_4 x_5 x_4 x_9 x_{10} \mathtt{b} x_{11} \mathtt{a} x_{12} \mathtt{b} x_{13} \mathtt{a} x_{14} x_{15} \mathtt{b} x_{15}^2 x_{16}^2 \mathtt{b} x_{17}$$

$$x_1 \mathtt{ab} x_6 x_7^2 x_8 \mathtt{ab} x_{11} \mathtt{a} x_{12} \mathtt{b} x_{13} \mathtt{a} x_{14} x_{15} \mathtt{b} x_{15}^2 x_{16}^2 \mathtt{b} x_{17}$$

$$x_1 \mathtt{ab} x_6 x_7^2 x_8 \mathtt{ab} x_{11} \mathtt{a} x_{12} \mathtt{b} x_{13} \mathtt{a} x_{14} \mathtt{b}(\mathtt{aa})^* \mathtt{b} x_{17}$$

Example

$$x_1 a x_2 x_3^2 b x_4 x_3 x_5 x_6 x_7^2 x_8 x_9 x_5 x_3 a x_4 x_5 x_4 x_9 x_{10} b x_{11} a x_{12} b x_{13} a x_{14} x_{15} b x_{15}^2 x_{16}^2 b x_{17}$$

$$x_1 ab x_6 x_7^2 x_8 x_9 x_5 x_3 a x_4 x_5 x_4 x_9 x_{10} b x_{11} a x_{12} b x_{13} a x_{14} x_{15} b x_{15}^2 x_{16}^2 b x_{17}$$

$$x_1 ab x_6 x_7^2 x_8 ab x_{11} a x_{12} b x_{13} a x_{14} x_{15} b x_{15}^2 x_{16}^2 b x_{17}$$

$$x_1 ab x_6 x_7^2 x_8 ab x_{11} a x_{12} b x_{13} a x_{14} b (aa)^* b x_{17}$$

Example

$$x_1 \mathtt{a} x_2 x_3^2 \mathtt{b} x_4 x_3 x_5 x_6 x_7^2 x_8 x_9 x_5 x_3 \mathtt{a} x_4 x_5 x_4 x_9 x_{10} \mathtt{b} x_{11} \mathtt{a} x_{12} \mathtt{b} x_{13} \mathtt{a} x_{14} x_{15} \mathtt{b} x_{15}^2 x_{16}^2 \mathtt{b} x_{17}$$

$$x_1 \mathtt{ab} x_6 x_7^2 x_8 x_9 x_5 x_3 \mathtt{a} x_4 x_5 x_4 x_9 x_{10} \mathtt{b} x_{11} \mathtt{a} x_{12} \mathtt{b} x_{13} \mathtt{a} x_{14} x_{15} \mathtt{b} x_{15}^2 x_{16}^2 \mathtt{b} x_{17}$$

$$x_1 \mathtt{ab} x_6 x_7^2 x_8 \mathtt{ab} x_{11} \mathtt{a} x_{12} \mathtt{b} x_{13} \mathtt{a} x_{14} x_{15} \mathtt{b} x_{15}^2 x_{16}^2 \mathtt{b} x_{17}$$

$$x_1 \mathtt{ab} x_6 x_7^2 x_8 \mathtt{ab} x_{11} \mathtt{a} x_{12} \mathtt{b} x_{13} \mathtt{a} x_{14} \mathtt{b} (\mathtt{aa})^* \mathtt{b} x_{17}$$

Example

$$x_1\,\mathtt{a}\,x_2\,x_3^2\,\mathtt{b}\,x_4\,x_3\,x_5\,x_6\,x_7^2\,x_8\,x_9\,x_5\,x_3\,\mathtt{a}\,x_4\,x_5\,x_4\,x_9\,x_{10}\,\mathtt{b}\,x_{11}\,\mathtt{a}\,x_{12}\,\mathtt{b}\,x_{13}\,\mathtt{a}\,x_{14}\,x_{15}\,\mathtt{b}\,x_{15}^2\,x_{16}^2\,\mathtt{b}\,x_{17}$$

Example

$x_1\,\mathtt{a}\,x_2\,x_3^2\,\mathtt{b}\,x_4\,x_3\,x_5\,x_6\,x_7^2\,x_8\,x_9\,x_5\,x_3\,\mathtt{a}\,x_4\,x_5\,x_4\,x_9\,x_{10}\,\mathtt{b}\,x_{11}\,\mathtt{a}\,x_{12}\,\mathtt{b}\,x_{13}\,\mathtt{a}\,x_{14}\,x_{15}\,\mathtt{b}\,x_{15}^2\,x_{16}^2\,\mathtt{b}\,x_{17}$

Example

$x_1 \mathtt{a} x_2 x_3^2 \mathtt{b} x_4 x_3 x_5 x_6 x_7^2 x_8 x_9 x_5 x_3 \mathtt{a} x_4 x_5 x_4 x_9 x_{10} \mathtt{b} x_{11} \mathtt{a} x_{12} \mathtt{b} x_{13} \mathtt{a} x_{14} x_{15} \mathtt{b} x_{15}^2 x_{16}^2 \mathtt{b} x_{17}$

$x_1 \mathtt{a} x_2 \mathtt{b} \mathtt{a} x_{10} \mathtt{b} x_{11} \mathtt{a} x_{12} \mathtt{b} x_{13} \mathtt{a} x_{14} x_{15} \mathtt{b} x_{15}^2 x_{16}^2 \mathtt{b} x_{17}$

Example

$x_1 \mathtt{a} x_2 x_3^2 \mathtt{b} x_4 x_3 x_5 x_6 x_7^2 x_8 x_9 x_5 x_3 \mathtt{a} x_4 x_5 x_4 x_9 x_{10} \mathtt{b} x_{11} \mathtt{a} x_{12} \mathtt{b} x_{13} \mathtt{a} x_{14} x_{15} \mathtt{b} x_{15}^2 x_{16}^2 \mathtt{b} x_{17}$

$x_1 \mathtt{a} x_2 \mathtt{b} \mathtt{a} x_{10} \mathtt{b} x_{11} \mathtt{a} x_{12} \mathtt{b} x_{13} \mathtt{a} x_{14} x_{15} \mathtt{b} x_{15}^2 x_{16}^2 \mathtt{b} x_{17}$

Example

$$x_1 a x_2 x_3^2 b x_4 x_3 x_5 x_6 x_7^2 x_8 x_9 x_5 x_3 a x_4 x_5 x_4 x_9 x_{10} b x_{11} a x_{12} b x_{13} a x_{14} x_{15} b x_{15}^2 x_{16}^2 b x_{17}$$

$$x_1 a x_2 b a x_{10} b x_{11} a x_{12} b x_{13} a x_{14} x_{15} b x_{15}^2 x_{16}^2 b x_{17}$$

Example

$$x_1 \mathtt{a} x_2 x_3^2 \mathtt{b} x_4 x_3 x_5 x_6 x_7^2 x_8 x_9 x_5 x_3 \mathtt{a} x_4 x_5 x_4 x_9 x_{10} \mathtt{b} x_{11} \mathtt{a} x_{12} \mathtt{b} x_{13} \mathtt{a} x_{14} x_{15} \mathtt{b} x_{15}^2 x_{16}^2 \mathtt{b} x_{17}$$

$$x_1 \mathtt{a} x_2 \mathtt{b} \mathtt{a} x_{10} \mathtt{b} x_{11} \mathtt{a} x_{12} \mathtt{b} x_{13} \mathtt{a} x_{14} x_{15} \mathtt{b} x_{15}^2 x_{16}^2 \mathtt{b} x_{17}$$

$$x_1 \mathtt{a} x_2 \mathtt{b} \mathtt{a} x_{10} \mathtt{b} x_{11} \mathtt{a} x_{12} \mathtt{b} x_{13} \mathtt{a} x_{14} \mathtt{b} (\mathtt{aa})^* \mathtt{b} x_{17}$$

Example

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$$x_1 a x_2 b a x_{10} b x_{11} a x_{12} b x_{13} a x_{14} x_{15} b x_{15}^2 x_{16}^2 b x_{17}$$

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Example

$x_1 a x_2 x_3^2 b x_4 x_3 x_5 x_6 x_7^2 x_8 x_9 x_5 x_3 a x_4 x_5 x_4 x_9 x_{10} b x_{11} a x_{12} b x_{13} a x_{14} x_{15} b x_{15}^2 x_{16}^2 b x_{17}$

$x_1 a x_2 b a x_{10} b x_{11} a x_{12} b x_{13} a x_{14} x_{15} b x_{15}^2 x_{16}^2 b x_{17}$

$x_1 a x_2 b a x_{10} b x_{11} a x_{12} b x_{13} a x_{14} b (aa)^* b x_{17}$

$\Rightarrow L_{E,\{a,b\}}(x_1 a x_2 b a x_{10} b x_{11} a x_{12} b x_{13} a x_{14} b (aa)^* b x_{17}) \in \text{REG}.$

Conclusion

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Patterns (with repeated variables) describe generalisations of the copy language.

If the corresponding terminal alphabet is small, then the complex "copy" structure may collapse.

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The copy language is non-regular and non-context-free, ...

...but we have to be careful with "copy-like" languages.